A FUNCTIONAL CALCULUS IN HILBERT SPACE BASED ON OPERATOR VALUED ANALYTIC FUNCTIONS(1)

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ABSTRACT

A homomorphic map is defined from the algebra of norm bounded analytic N-operator valued functions in the unit disc into the algebra of bounded operators in Hilbert spaces represented as left invariant subspaces of $H^2(N)$, and the spectral properties of the map are studied.
The subclass of functions having norm bound one in the disc is characterized in terms of the power series coefficients.

1. **Introduction.** The object of this paper is the construction of a functional calculus for a class of contraction operators in Hilbert space in which the functions involved are elements of certain algebras of bounded operator valued analytic functions in the unit disc. This generalizes a calculus constructed by Sz.-Nagy and Foias [11] in which the function algebra used was H^∞. The class of operators for which this construction is possible is the set of all contractions T in a separable Hilbert space H for which T^{*n} converges to zero in the strong operator topology. By a theorem of Rota [9, 2, 12] we can represent T^* as the restriction of the left shift operator to a left invariant subspace of a direct sum, finite or infinite, of H^2 spaces. Therefore we will assume the operators to have been so represented. In §2 we introduce the functional calculus and study its spectral properties. As in [4] the tools are the Beurling-Lax representation of right invariant subspaces and a matrix version of the corona theorem obtained by the author in [4]. In §3 we use the functional calculus to generalize a theorem of I. Schur [10] characterizing analytic operator valued functions of norm bound one in the unit disc in terms of the coefficients of their power series expansion. Before proceeding we will introduce some notation.

Let N be a separable Hilbert space. $H^2(N)$ is the Hardy class of order 2, i.e. the set of all N-valued square integrable functions on the unit circle whose Fourier coefficients vanish for all negative indices. (For details we refer to [6].) The $H^2(N)$ norm is defined by

$$\| F \| = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \| F(e^{it}) \|^2 dt \right\}^{1/2}.$$

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All functions in $H^2(N)$ have analytic continuations into the disc, and whenever convenient we will assume that the functions have been continued.

As usual a subspace of $H^2(N)$ is called (right) invariant if it is invariant under multiplication by z. The operator U of multiplication by z in $H^2(N)$ is called the right shift, i.e.

$$(UF)(z) = zF(z).$$

The adjoint of U, U^*, is called the left shift and we have

$$(U^*F)(z) = [F(z) - F(0)]/z.$$

A subspace of $H^2(N)$ is called left invariant if it is invariant under the left shift. The orthogonal complement of a left invariant subspace is right invariant.

Now let K be a left invariant subspace of $H^2(N)$ and P the orthogonal projection of $H^2(N)$ onto K. If we embed $H^2(N)$ naturally in $L^2(N)$, then we consider P to be the orthogonal projection of $L^2(N)$ onto K. Let us define the operator T in K by

$$TF = P(UF) \tag{1.1}$$

for each F in K. Clearly $T^* = U^* \mid K$. The operator T will be the basis of our calculus.

By the Beurling-Lax Theorem [1,7,5] the orthogonal complement of K in $H^2(N)$, $K^\perp$, is of the form $K^\perp = SH^2(N)$ where S is a rigid function i.e. an N-contraction valued analytic function in the unit disc having a.e. partial isometries with a fixed initial space as boundary values on the unit circle. A rigid function is called inner if its boundary values are a.e. unitary operators.

2. **Spectral analysis.** By $OH^\infty(N)$, (the O stands for operator), we denote the Banach algebra of all bounded N-operator valued analytic functions in the open unit disc D, the norm being given by

$$\| A \|_\infty = \sup\{\| A(z) \| \mid z \in D\}. \tag{2.1}$$

Here $\| A(z) \|$ is the norm of $A(z)$ as an operator in N. The algebra $OH^\infty(N)$ has a natural involution defined by $A \to \tilde{A}$, $\tilde{A}$ being defined by

$$\tilde{A}(z) = A(\bar{z})^*. \tag{2.2}$$

Clearly this involution satisfies besides the standard properties of an involution also

$$\| \tilde{A} \|_\infty = \| A \|_\infty. \tag{2.3}$$

If we define for each F in $H^2(N)$ and A in $OH^\infty(N)$

$$(A(U)F)(z) = A(z)F(z) \tag{2.4}$$

then the map $A \to A(U)$ is a norm preserving algebra homomorphism of $OH^\infty(N)$ into the algebra of all bounded operators in $H^2(N)$. Of course the identity function

z is mapped into the right shift U and this motivated our notation. Since for any A in $OH^{\infty}(N)$ we have $zA(z) = A(z)z$, it follows that $UA(U) = A(U)U$ or that the multiplication operators $A(U)$ all commute with U. The converse is also true and is given by the following simple generalization of the classical scalar theorem. In an equivalent form it is given in [8].

THEOREM 2.1. *Every bounded operator $\mathfrak{A}$ in $H^2(N)$ that commutes with the right shift U can be represented in the form $A(U)$, with A in $OH^{\infty}(N)$ and moreover*

$$\|\mathfrak{A}\| = \|A\|_{\infty}.$$

The map $A \to A(U)$ of $OH^{\infty}(N)$ into the algebra of all bounded operators in $H^2(N)$ can be greatly generalized. Instead of $H^2(N)$ we consider a left invariant subspace K of $H^2(N)$. By $OH^{\infty}_K(N)$ we denote the subalgebra of all $OH^{\infty}(N)$ functions that leave $K^{\perp}$ invariant. Since $K^{\perp}$ is a right invariant subspace we have $H^{\infty} \subset OH^{\infty}_K(N)$, where H^{∞} is considered as naturally embedded in $OH^{\infty}(N)$. The next definition introduces the functional calculus.

DEFINITION 2.1. *Let A be in $OH^{\infty}_K(N)$. For each F in K we define*

$$A(T)F = P(AF). \tag{2.5}$$

Obviously this generalizes the Sz.-Nagy-Foias calculus as applied to the operator T. The next theorem summarizes the elementary properties of this calculus.

THEOREM 2.2. *$A, B \in OH^{\infty}_K(N)$, $\alpha, \beta \in C$ then*

a. $(\alpha A + \beta B)(T) = \alpha A(T) + \beta B(T)$

b. $(AB)(T) = A(T)B(T)$

c. $\|A(T)\| \leqq \|A\|_{\infty}$

d. *$A_n, A \in OH^{\infty}_K(N)$. Let $\{A_n\}$ be uniformly bounded in D and such that* a.e. *on the unit circle lim $A_n(e^{it}) = A(e^{it})$ in the strong operator topology of N; then $A_n(T)$ converges to $A(T)$ in the strong operator topology in K.*

Proof.

a. Obvious

b. Here we use the fact that multiplication by A leaves K invariant. It should be noted that *b.* holds even if B is only in $OH^{\infty}(N)$.

c. For each F in K,

$$\|A(T)F\|^2 \leqq \frac{1}{2\pi}\int_0^{2\pi} \|A(e^{it})F(e^{it})\|^2 dt \leqq \frac{1}{2\pi}\int_0^{2\pi} \|A\|_{\infty}^2 \|F(e^{it})\|^2 dt$$

$$= \|A\|_{\infty}^2 \|F\|^2.$$

The first inequality follows from the fact that a projection is not norm increasing.

d. Let F be in k,

$$\| (A_n(T) - A(T))F \|^2 \leqq \frac{1}{2\pi} \int_0^{2\pi} \| (A_n(e^{it}) - A(e^{it}))F(e^{it}) \|^2 dt$$

and we use the Lebesgue convergence theorem.

While the calculus is well defined for all left invariant subspaces of $H^2(N)$ we will restrict ourselves as of now to those subspaces for which the Beurling Lax representation of their orthogonal complement can be given by an inner function. Thus we assume $K = H^2(N) \ominus SH^2(N)$ where S is inner.

LEMMA 2.1. *For A in $OH^\infty_K(N)$, $A(T) = 0$ if and only if there exists a C in $OH^\infty(N)$ satisfying*

$$A = SC. \tag{2.6}$$

Proof. If there exists a C in $OH^\infty(N)$ for which (2.6) holds, then for each $F \in K$, $AF = SCF$ is in $K^\perp$. Thus $A(T)f = 0$.

Conversely suppose $A(T) = 0$. Since $A \in OH^\infty_K(N)$, this means that the range of A as a multiplication operator in $H^2(N)$ is included in $SH^2(N)$. Thus for each F in $H^2(N)$ the operator C defined by

$$(CF)(e^{it}) = S(e^{it})^* A(e^{it}) F(e^{it}) \tag{2.7}$$

is a bounded operator in $H^2(N)$ that clearly commutes with the right shift. By Theorem 2.1, C is in $OH^\infty(N)$ and (2.6) is proved.

THEOREM 2.3. *Let $K = H^2(N) \ominus SH^2(N)$ with S inner. $A \in OH^\infty_K(N)$ if and only if $A \in OH^\infty(N)$ and there exists an A_1 in $OH^\infty(N)$ satisfying*

$$A(z)S(z) = S(z)A_1(z) \tag{2.8}$$

for all z in D. If such an A_1 exists then it is unique.

Proof. If (2.8) is satisfied, then clearly A leaves $SH^2(N)$ invariant as for each $F \in H^2(N)$

$$A(SF) = S(A_1 F) \in SH^2(N).$$

Conversely assume $A \in OH^\infty_K(N)$. For each $F \in H^2(N)$, $ASF \in SH^2(N)$ thus $S^*ASF \in H^2(N)$. It follows that the operator A_1 in $H^2(N)$ defined by

$$(A_1 F)(e^{it}) = S(e^{it})^* A(e^{it}) S(e^{it}) F(e^{it}) \tag{2.9}$$

is a bounded operator commuting with the right shift. Hence by Theorem 4.2, $A_1 \in OH^\infty(N)$, and

$$(A_1 F)(e^{it}) = A_1(e^{it}) F(e^{it}). \tag{2.10}$$

But (2.9) and (2.10) imply (2.8) and the theorem is proved. The uniqueness follows from Lemma 2.1.

The next two theorems are concerned with the representations of $A(T)^*$ as a multiplication operator. This is possible if the rigid function corresponding to K is inner. Thus again we assume that $K = H^2(N) \ominus SH^2(N)$ where S is inner. Obviously $\tilde{S}$ defined by (2.2) is inner too. Let us define a left invariant subspace $\tilde{K}$ of $H^2(N)$ by

$$\tilde{K} = H^2(N) \ominus \tilde{S}H^2(N). \tag{2.11}$$

Let $A \in OH^\infty_K(N)$. By Theorem 2.3 there exists a unique A_1 such that (2.8) is satisfied.

THEOREM 2.4. *Let A_1 be given by* (2.8), *then $\tilde{A}_1$ as defined by* (2.2) *is in $OH^\infty_{\tilde{K}}(N)$.*

Proof. By Theorem 2.3, $\tilde{A}_1$ is in $OH^\infty_{\tilde{K}}(N)$ if and only if there exists a B in $OH^\infty(N)$ for which

$$\tilde{A}_1(z)\tilde{S}(z) = \tilde{S}(z)\tilde{B}(z). \tag{2.12}$$

But for $B = A$ this is clearly equivalent to (2.8).

Thus $\tilde{A}_1$ leaves $\tilde{K}^\perp$ invariant. Let us denote by $\tilde{T}$ the operator in $\tilde{K}$ of multiplication by z followed by projection into $\tilde{K}$. Let $P, \tilde{P}$ be the orthogonal projections of $H^2(N)$ (or $L^2(N)$) onto $K, \tilde{K}$ respectively. For $A \in OH^\infty_{\tilde{K}}(N)$ we denote by $A(\tilde{T})$ the operator given by

$$A(\tilde{T})F = \tilde{P}(AF). \tag{2.13}$$

THEOREM 2.5. *Let A_1 be given by* (2.8); *then $\tilde{A}_1(\tilde{T})$ is unitarily equivalent to $A(T)^*$.*

Proof. We have the following direct sum decompositions,

$$L^2(N) = K^2(N) \oplus K \oplus SH^2(N) = K^2(N) \oplus \tilde{K} \oplus \tilde{S}H^2(N).$$

We define the map τ in $L^2(N)$ by

$$(\tau F)(e^{it}) = e^{-it}S(e^{-it})^*F(e^{-it}). \tag{2.14}$$

τ is a unitary map of $L^2(N)$, mapping $SH^2(N)$ onto $K^2(N)$, $K^2(N)$ onto $\tilde{S}H^2(N)$ and K onto $\tilde{K}$. Now for $F \in K$

$$\begin{aligned}(\tau AF)(e^{it}) &= e^{-it}S(e^{-it})^*A(e^{-it})F(e^{-it})\\ &= e^{-it}\tilde{S}(e^{it})\tilde{A}(e^{it})^*F(e^{-it}).\end{aligned}$$

τ being unitary, we have for all F, G in K

$$\begin{aligned}(\tau AF,\tau G) &= (AF,G)\\ &= \frac{1}{2\pi}\int_0^{2\pi}(e^{-it}\tilde{S}(e^{it})\tilde{A}(e^{it})^*F(e^{-it}),e^{-it}\tilde{S}(e^{it})G(e^{it}))\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}(F(e^{-it}),\tilde{A}(e^{it})G(e^{-it}))\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}(e^{-it}\tilde{S}(e^{it})F(e^{-it}),e^{-it}\tilde{S}(e^{it})\tilde{A}(e^{it})G(e^{-it}))\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}(e^{-it}\tilde{S}(e^{it})F(e^{-it}),e^{-it}\tilde{A}_1(e^{it})\tilde{S}(e^{it})G(e^{-it}))\,dt\\ &= (\tau F,\tilde{A}_1\tau G).\end{aligned}$$

Taken together with the relation

$$\tau P = \tilde{P}\tau \tag{2.15}$$

this gives

$$(\tau PAF,\tau G) = (\tilde{P}\tau AF,\tau G) = (\tau F,\tilde{P}\tilde{A}_1\tau G)$$

or $A(T)^*$ is unitarily equivalent to $\tilde{A}_1(\tilde{T})$.

If we consider the special case of the function $A(z) = zI$ then under the same assumptions we get the unitary equivalence of $\tilde{T}$ and T^*. This is Theorem 2.1 in [4].

DEFINITION 2.2. *Let A be in $OH^\infty(N)$. An inner function R in $OH^\infty(N)$ is called a left inner factor of A if there exists a B in $OH^\infty(N)$ such that $A = RB$. R is a right inner factor of A if $\tilde{R}$ is a left inner factor of $\tilde{A}$.*

LEMMA 2.2. *Let $K = H^2(N)\ominus SH^2(N)$, S inner, $A\in OH_K(N)$. A and S have no non-trivial common left inner factor if and only if the manifold*

$$A(T)K = \{PAF \mid F\in K\}$$

is dense in K.

Proof. If A and S have a non-trivial common left inner factor then there exists a non-null F in K such that F is orthogonal to $AH^2(N)$. Hence F is orthogonal to $PAH^2(N)$ i.e. to $A(T)K$.

Conversely if A and S have no non-trivial common left inner factor then $AH^2(N) + SH^2(N)$ spans $H^2(N)$. Therefore for each F in K for which $(F,A(T)G) = 0$ is true for all G in K, the more general orthogonality condition $(F,H) = 0$ for all $H\in H^2(N)$ is also satisfied. Hence $F = 0$. It follows that $A(T)K$ is dense in K.

THEOREM 2.6. *$0\in\sigma_p(A(T)^*)$ if and only if A,S have a non-trivial common left inner factor.*

Proof. By Lemma 2.2 A and S have a non-trivial common left inner factor if and only if $A(T)K$ is not dense in K which is equivalent to $0 \in \sigma_p(A(T)^*)$.

Again let $A \in OH_K^\infty(N)$. By Theorem 2.4 there exists an $A_1 \in OH_K^\infty(N)$ for which (2.8) holds.

THEOREM 2.7. *$0 \in \sigma_p(A(T))$ if and only if S and A_1 have a non-trivial right inner factor.*

Proof. By Theorem 2.5 $A(T)$ is unitarily equivalent to $\tilde{A}_1(\tilde{T})^*$. Applying Theorem 2.6 to $\tilde{A}_1(\tilde{T})^*$ we have $0 \in \sigma_p(\tilde{A}_1(\tilde{T})^*)$ if and only if $\tilde{A}_1$ and $\tilde{S}$ have a non-trivial common left inner factor i.e. if and only if A_1 and S have a non-trivial common right inner factor. But $0 \in \sigma_p(A(T))$ if and only if $0 \in \sigma_p(\tilde{A}_1(\tilde{T})^*)$ and this completes the proof.

In order to get results about the invertibility of the operators $A(T)$ we will have to assume the auxiliary Hilbert space N to be finite dimensional. In that case we have available to us a matrix generalization of the Carleson Corona Theorem [4] which we proceed to quote.

THEOREM 2.8. *Let N be an n dimensional Hilbert space, A_i, $i = 1, \cdots, p$ in $OH^\infty(N)$.*

a. *A necessary and sufficient condition for the existence of B_i in $OH^\infty(N)$ such that $\sum_{i=1}^n B_i(z)A_i(z) = I$ is the existence of a $\delta > 0$ for which*

$$\inf \left\{ \sum_{i=1} \| A_i(z)x \| \;\middle|\; x \text{ in } N, \ \| x \| = 1 \right\} \geqq \delta \tag{2.16}$$

for all z in D.

b. *A necessary and sufficient condition for the existence of B_i in $OH^\infty(N)$ such that $\sum_{i=1} A_i(z) B_i(z) = I$ is the existence of a $\delta > 0$ for which*

$$\inf \left\{ \sum_{i=1} \| A_i(z)^* x \| \;\middle|\; x \text{ in } N, \ \| x \| = 1 \right\} \geqq \delta \tag{2.17}$$

for all z in D.

THEOREM 2.9. *Let $K = H^2(N) \ominus SH^2(N)$ where S in an inner function and N a finite dimensional Hilbert space. $A(T)$ as defined by (2.5) has a bounded inverse if and only if there exists a $\delta > 0$ for which*

$$\inf \{ \| A(z)^* x \| + \| S(z)^* x \| \mid x \text{ in } N, \ \| x \| = 1 \} \geqq \delta \tag{2.18}$$

and

$$\inf \{ \| A_1(z) x \| + \| S(z) x \| \mid x \text{ in } N, \ \| x \| = 1 \} \geqq \delta \tag{2.19}$$

for all z in D, A_1 as defined by (2.8).

Proof. We will show first that under the assumption that no $\delta > 0$ exists for which (2.19) holds, there exists a sequence of functions F_n in K for wgich

$$\lim \| F_n \| = 1 \text{ and } \lim \| A(T)F_n \| = 0,$$

which implies that $A(T)$ has no bounded inverse.

By our assumption there is a sequence $\{\lambda_n\}$ of points in D and unit vectors x_n in N for which

$$\lim \| A_1(\lambda_n)x_n \| = \lim \| S(\lambda_n)x_n \| = 0.$$

If $S(\lambda)x = A_1(\lambda)x = 0$ then S and A_1 have a common right inner factor and by Theorem 2.7, $0 \in \sigma_p(A(T))$. In fact in this case we can easily exhibit a null function of $A(T)$. By Theorem 2.2 in [4], $S(\lambda)x = 0$ implies that $F(z) = \frac{S(z)x}{z - \lambda}$ is in K and is moreover an eigenfunction of T corresponding to the eigenvalue λ. Now

$$A(z)F(z) = A(z)\frac{S(z)x}{z-\lambda} = \frac{S(z)A_1(z)x}{z-\lambda}.$$

But since $A_1(\lambda)x = 0$, $\frac{A_1(z)x}{z-\lambda}$ is in $H^2(N)$, and $S(z)\frac{A_1(z)x}{z-\lambda}$ is in $SH^2(N)$. It follows that AF is in $SH^2(N)$ and thus $A(T)F = 0$. In general we have no eigenfunctions but approximate ones.

$$\begin{aligned}(1 - |\lambda_n|^2)^{\frac{1}{2}} &\frac{S(z)x_n}{z-\lambda_n} \\ &= (1 - |\lambda_n|^2)^{\frac{1}{2}}\frac{S(\lambda_n)x_n}{z-\lambda_n} + (1-|\lambda_n|^2)^{\frac{1}{2}}\frac{(S(z)-S(\lambda_n))x_n}{z-\lambda_n}.\end{aligned}$$

Since the function $\frac{S(z)x_n}{z-\lambda_n}$ is orthogonal to $SH^2(N)$ its projection into $H^2(N)$ is also orthogonal to $SH^2(N)$. Let

$$F_n(z) = (1-|\lambda_n|^2)^{1/2}\frac{(S(z)-S(\lambda_n))x_n}{z-\lambda_n}$$

and

$$G_n(z) = (1-|\lambda_n|^2)^{\frac{1}{2}}\frac{S(\lambda_n)x_n}{z-\lambda_n}.$$

Clearly G_n is conjugate analytic and F_n analytic, thus F_n is in K. Moreover we have $\lim \| G_n \| = \lim \| S(\lambda_n)x_n \| = 0$ and thus $\lim \| F_n \| = 1$. Now

$$\begin{aligned}A(z)F_n(z) &= A(z)(1-|\lambda_n|^2)^{\frac{1}{2}}\frac{(S(z)-S(\lambda_n))x_n}{z-\lambda_n} \\ &= S(z)(1-|\lambda_n|^2)^{\frac{1}{2}}\frac{A_1(z)x_n}{z-\lambda_n} - A(z)(1-|\lambda_n|^2)^{\frac{1}{2}}\frac{S(\lambda_n)x_n}{z-\lambda_n} \\ &= S(z)(1-|\lambda_n|^2)^{1/2}\frac{(A_1(z)-A_1(\lambda_n))x_n}{z-\lambda_n} \\ &\quad + S(z)(1-|\lambda_n|^2)^{\frac{1}{2}}\frac{A_1(\lambda_n)x_n}{z-\lambda_n} - A(z)(1-|\lambda_n|^2)^{\frac{1}{2}}\frac{S(\lambda_n)x_n}{z-\lambda_n}\end{aligned}$$

The first term on the right is in $SH^2(N)$ and thus its projection into K is zero. The following estimate follows.

$$\begin{aligned}\|A(T)F_n\| &< \|PAF_n\| \\ &\leqq (1-|\lambda_n|^2)^{\frac{1}{2}} \left\{\frac{1}{2\pi}\int_0^{2\pi} \left\|\frac{S(e^{it})A_1(\lambda_n)x_n}{e^{it}-\lambda_n}\right\|^2 dt\right\}^{\frac{1}{2}} \\ &\quad + (1-|\lambda_n|^2)^{\frac{1}{2}} \left\{\frac{1}{2\pi}\int_0^{2\pi} \left\|\frac{A(e^{it})S(\lambda_n)x_n}{e^{it}-\lambda_n}\right\|^2 dt\right\}^{\frac{1}{2}} \\ &\leqq \|A_1(\lambda_n)x_n\| + \|A\|_\infty \|S(\lambda_n)x_n\|.\end{aligned}$$

Hence $\lim A(T)F_n = 0$.

Next we consider the necessity of (2.18). If $A(T)$ has a bounded inverse so has $A(T)^*$ which by Theorem 2.5 is unitary equivalent to $\tilde{A}_1(\tilde{T})$. Applying the former considerations to $\tilde{A}_1(\tilde{T})$ and remembering that $\tilde{A}_1\tilde{S} = \tilde{S}\tilde{A}$ we get (2.18).

To prove sufficiency let us assume there exists a $\delta > 0$ for which (2.18) and (2.19) hold. Theorem 2.8 and (2.18) imply the existence of $C, D \in OH^\infty(N)$ for which $C\tilde{A} + D\tilde{S} = 1$, or equivalently $A\tilde{C} + S\tilde{D} = 1$. By taking projections onto K and using the fact that $S(T)\tilde{D}(T) = 0$, we get

$$A(T)\tilde{C}(T) = 1. \tag{2.20}$$

(In this connection, see the proof of Theorem 2.2b).

By the same token from (2.19) follows the existence of $C_1, D_1 \in OH^\infty(N)$ for which

$$C_1A_1 + D_1S = 1 \text{ or } \tilde{A}_1\tilde{C}_1 + \tilde{S}\tilde{D}_1 = 1.$$

By taking the projection onto K we get

$$\tilde{A}_1(\tilde{T})\tilde{C}_1(\tilde{T}) = 1. \tag{2.21}$$

Equation (2.20) says that $A(T)$ has a bounded right inverse. Equation (2.21) says that $\tilde{A}_1(\tilde{T})$, and thus by Theorem 2.5 also $A(T)^*$, has a bounded right inverse. By taking adjoints this is equivalent to $A(T)$ having a bounded left inverse. Thus $A(T)$ has bounded right and left inverses and is thus boundedly invertible.

REMARK. In the proof of the necessity of conditions (2.18) and (2.19) the finite dimensionality of N has not been used and thus that part of the theorem holds in the general case.

COROLLARY 2.1. *Let u be in H^∞; $u(T)$ has a bounded inverse if and only if there exists a $\delta > 0$ for which*

$$|u(z)| + \|S(z)^{-1}\|^{-1} \geqq \delta$$

for all z in D. $\| S(z)^{-1}\|^{-1}$ *has to be interpreted as zero wherever* $S(z)$ *is not invertible.*

Proof. Since $u(z)S(z) = S(z)u(z)$ we have $u_1(z) = u(z)$. Condition (2.18) reduces, since u is a scalar function, to

$$|u(z)| + \inf_{\|x\|=1} \| S(z)^*x \| \geqq \delta.$$

But if $S(z)^*$ is invertible $\inf \| S(z)^*x \| = \| S(z)^{*-1} \|^{-1} = \| S(z)^{-1} \|^{-1}$. Otherwise N being finite dimensional there is a null vector and $\inf \| S(z)^*x \| = 0$. Condition (2.19) reduces similarly. Thus we get Theorem 2.3 in [4] as a special case.

3. **Analytic operator valued functions of bound one in the unit disc.** This section is devoted to a generalization of a theorem of I. Schur [10] characterizing analytic functions of bound one in the unit disc in terms of their power series coefficients. Our approach via Hilbert space operator theoretic methods is very simple and is close in spirit to Schur's own proof though it differs considerably in language, Schur's papers preceding of course the abstract formulation of Hilbert space.

Let N be a separable Hilbert space. We denote by S_N (S for Schur) the class of all N-contraction valued analytic functions in the unit disc D; i.e. A is in S_N if A is in $OH^\infty(N)$ and $\| A \|_\infty \leqq 1$.

Let $K_n = H^2(N) \ominus z^{n+1}H^2(N)$ and T_n multiplication by z in K_n followed by projection into K_n. Since z^n are scalar inner functions we have $OH^\infty_{K_n}(N) = OH^\infty(N)$. $A(T_n)$ is defined by (2.5)

THEOREM (3.1). *Let* $A(z) = \sum_{n=0}^\infty A_n z^n$ *be an N–operator valued analytic function.* $A \in S_N$ *if and only if the quadratic form*

$$P_{ij} = \delta_{ij} - \sum_{v=0}^{\min(i,j)} A_{j-v}A^*_{i-v} \tag{3.1}$$

is non-negative definite.

Proof. Assume $A \in S_N$; it follows that for each n, $\| A(T_n) \| \leqq 1$. Now $T_n^{n+1} = 0$ and

$$A(T_n) = A_0 + A_1 T_n + \cdots + A_n T_n^n. \tag{3.2}$$

It should be noted that in this case T_n and A_i are commuting operators resulting from the fact that the z^n are scalar. In general we cannot expect relation (3.2). Taking adjoints and using the commutativity noted above we have

$$A(T_n)^* = A_0^* + A_1^*T_n^* + \cdots + A_n^*T_n^{*n}.$$

Now

$$K_n = \{F(z) = x_0 + \cdots + x_n z^n \,|\, x_i \in N\}.$$

$$\begin{aligned}(A(T_n)^*F)(z) &= A_0^*(x_0 + \cdots + x_n z^n) + \cdots + A_n^* x_n \\ &= (A_0^*x_0 + \cdots + A_n^*x_n) + \cdots + A_0^*x_n z^n.\end{aligned}$$

Since $\| A(T_n)^* \| = \| A(T_n) \| < 1$, we have for each $F \in K_n$, $\| A(T_n)^* F \|^2 \leqq \| F \|^2$. Expressed in terms of the K_n norm we have for all $x_i \in N$

$$\| A_0^* x_0 + \cdots + A_n^* x_n \|^2 + \cdots + \| A_0^* x_n \|^2 \leqq \| x_0 \|^2 + \cdots + \| x_n \|^2. \tag{3.3}$$

Equivalently $\sum_{i,j=0}^n (P_{ij} x_i, x_j) \geqq 0$ for all $x_i \in N$, where P_{ij} is as in (3.1). Since n is arbitrary, the non-negative definiteness of the form (3.1) is proved.

Conversely we assume now that the form (3.1) is non-negative definite. In K_n we define an operator $\mathfrak{A}_n$ by

$$\mathfrak{A}_n^*(x_0 + \cdots + x_n z^n) = A_0^*(x_0 + \cdots + x_n z^n) + A_1^*(x_1 + \cdots + x_n z^{n-1}) + \cdots + A_n^* x_n.$$

By (3.3) $\mathfrak{A}_n$ is a contraction and it obviously commutes with T_n^*. For $n > m$ $\mathfrak{A}_m = \mathfrak{A}_n \mid K_m$. On the linear manifold $\bigcup_{n=0}^\infty K_n$, which is dense in $H^2(N)$ we define $\mathfrak{A}^*$ by

$$\mathfrak{A}^* F = \mathfrak{A}_n^* F \text{ for } F \in K_n.$$

$\mathfrak{A}^*$ is well defined and commutes with the left shift in $H^2(N)$. It extends by continuity to an everywhere defined contraction commuting with the left shift. This new operator we still denote by $\mathfrak{A}^*$. Taking adjoints, $\mathfrak{A} = \mathfrak{A}^{**}$, is a contraction commuting with multiplication by z in $H^2(N)$. By Theorem 2.1 $\mathfrak{A}$ can be represented as $(\mathfrak{A}F)(z) = A(z)F(z)$ with $A \in S_N$. Since $A(T_n) = \mathfrak{A}_n$ we must have $A(z) = \sum_{n=0}^\infty A_n z^n$, which proves the theorem.

Corollary 3.1 *Let* $A(z) = \sum_{n=0}^\infty A_n z^n$ *be an* N *operator valued analytic function.* $A \in S_N$ *if and only if the quadratic form*

$$Q_{ij} = \delta_{ij} - \sum_{\nu=0}^{\min(i,j)} A_{j-\nu}^* A_{i-\nu}$$

is non-negative definite.

Proof. $A \in S_N$ if and only if $\tilde{A} \in S_N$ where $\tilde{A}(z) = A(\bar{z})^* = \sum_{n=0}^\infty A_n^* z^n$. For $\tilde{A}$ we apply Theorem 3.1.

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